A GEOMETRIC REPRESENTATION.*

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(*Continued from Vol. X, page 210.*)

§ 4. The Surface on Which a Family of Spirals Lies.

Given the equation of the family of spirals

(1) $$y = F(x) e^{f(x, k) i}; \tag{12}$$

then by § 3, the equation of the surface on which the family lies is

$$y = F(x) e^{\phi i} \tag{13}$$

or $$y^2 + z^2 = (F(x))^2 \tag{14}$$

in rectangular co-ordinates, a surface of revolution.

(2) Given $y = F(x, \tan f(x, k), \operatorname{ctn} f(x, k)) e^{f(x, k) i}$

$$= H(x, \tan f(x, k)) e^{f(x, k) i}; \tag{15}$$

then the surface on which the family lies is

$$y = H(x, \tan \phi) e^{\phi i} \tag{16}$$

or in rectangular co-ordinates

$$y^2 + z^2 = \left(H\left(x, \frac{z}{y}\right)\right)^2. \tag{17}$$

This applied to $y = R(x, \phi) e^{\phi i}$ gives

$$y^2 + z^2 = \left(R\left(x, \frac{z}{y}\right)\right)^2, \tag{18}$$

which after reducing becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The difference between cases (1) and (2) is that $F(x)$ in (1) *may not be expressed* in the form $F(x) = H(x, \tan f(x, k))$. This will be true for every surface of revolution.

§ 5. The Length of an Arc of the Spiral on the Surface $f(x, y, z) = 0$.

If $F_1(x, f(x, k)) = f_1(x)$, $F_2(x, f(x, k)) = f_2(x)$, an arc s of the spiral is given by

$$s = \int (1 + (f_1'(x))^2 + (f_2'(x))^2)^{\frac{1}{2}} dx + C. \qquad (19)$$

Examples.

1. The length of one turn of the helix $y = ae^{nxi}$ (n a parameter) is

$$\begin{aligned} s &= \int_0^{2\pi/n} (1 + a^2n^2 \sin^2 nx + a^2n^2 \cos^2 nx)^{\frac{1}{2}} dx \\ &= \int_0^{2\pi/n} (1 + a^2n^2)^{\frac{1}{2}} dx = \left(\frac{1}{n^2} + a^2\right)^{\frac{1}{2}} 2\pi. \end{aligned} \qquad (20)$$

2. If $a = b = c$ in the ellipsoid $y = R(x, \phi)e^{\phi i}$, we get

$$y = (a^2 - x^2)^{\frac{1}{2}} e^{\phi i}, \qquad (21)$$

the equation of a sphere.

$$y = (a^2 - x^2)^{\frac{1}{2}} e^{kxi} \qquad (22)$$

is the equation of a particular family of spirals on the sphere where $f(x) = x$. The length of an arc of one of the spirals is

$$s = \int \left(\frac{a^2}{a^2 - x^2} + k^2(a^2 - x^2)\right)^{\frac{1}{2}} dx + c, \qquad (23)$$

reducible by $x = a \sin \phi$ and $k = 2n\pi/a$ to

$$s = a\int (1 + (2n\pi)^2 \cos^4 \phi)^{\frac{1}{2}} d\phi + C. \qquad (24)$$

§ 6. The Function $y = (x^{x-1}(x - 1)^x)^{1/1-2x}$.

I. *Preliminary Statement.*

The following is a statement of some of the results obtained in the investigation of this function.

For any value of x, y is in general multiple-valued.

There is an infinite number of p-valued y's ($p = 1, 2, 3 \cdots \infty$). When x is commensurable the number of values of y is finite, and infinite when x is incommensurable. All the values of

(x, y) for a given x are represented by points on the same circle with center in OX, perpendicular to OX, at distance x from O, and with radius equal to $|y|$. From $x = -\infty$ to $x = 0$ and from $x = 1$ to $x = \infty$ there is a real continuous curve in addition to an infinity of discrete real points. From $x = -\infty$ to $x = 0$, the continuous real values of y are negative, the discrete real values positive. From $x = 1$ to $x = \infty$ this is reversed. The axis of x is an asymptote, and for $\begin{matrix} x = 0 \\ x = 1 \end{matrix}$, $y = \infty$, and $\begin{matrix} x = 0 \\ x = 1 \end{matrix}$ are asymptotes. From $x = 0$ to $x = 1$, there is no continuous real curve but only discrete real positive and negative points and complex points. From $x = -\infty$ to $x = \infty$ there is an infinity of conjugate right- and left-handed spirals which pierce the real plane at the discrete real points or in other real points. The discrete real points lie infinitely close together, but so that between any two positive or any two negative real points however close, an infinity of both real and complex points still lies. The same is true for any two purely imaginary points. As x advances and the points representing y move on spirals the distribution of points on the circles may be best described as one of infinitely complex kaleidoscopic change. The spirals pass through the points on the circles, a point corresponding or belonging to a group of spirals, but in the case of an incommensurable x there is a one-to-one correspondence, one value belonging to one spiral and conversely. When y is single-valued it belongs to all spirals, which pass through it. From $x = 0$ to $x = \frac{1}{2}$ and from $x = 1$ to $x = \infty$ $|y|$ decreases, and from $x = \frac{1}{2}$ to $x = 1$ and from $x = -\infty$ to $x = 0$ $|y|$ increases. At $x = \frac{1}{2}$, $|y| = 2e = 5.4+$ approximately and $|y|$ is at a minimum. At $x = \infty$, $|y| = 0$, an absolute minimum. The whole representation both for all real and all complex points is symmetrical with respect to the point $(\frac{1}{2}, 0)$. The positive and negative discrete real points lie on the real curves of $y = \pm(x^{x-1}(1-x)^x)^{1/1-2x}$, the upper sign for the curve through the positive points. These real curves are symmetrical with respect to $(\frac{1}{2}, 0)$ and also with respect to the lines $y = 0$, $x = \frac{1}{2}$. When $x > 1$, these curves contain complex and discrete real points which lie on the real curves of $y = \pm(x^{x-1}(x-1)^x)^{1/1-2x}$. They also contain spirals. All

the real and complex points and spirals of all the curves mentioned lie on the surface of revolution

$$y = (x^{x-1}(1 - x)^x)^{1/(1-2x)}e^{\phi i} \tag{25}$$

or

$$y^2 + z^2 = (\mathrm{mod}\,(x^{x-1}(x - 1)^x)^{1/(1-2x)})^2. \tag{26}$$

II. *Properties and Special Values of the Function.*

1. *Symmetry.*—Transforming to (½, 0) as origin we have

$$y_x = ((x + \tfrac{1}{2})^{x-\frac{1}{2}}(x - \tfrac{1}{2})^{x+\frac{1}{2}})^{-(1/2x)}. \tag{27}$$

Changing x into $-x$, we find

$$y_{-x} = -y_x. \tag{28}$$

For the curves through the real discrete points between 0 and 1

$$y_x = \pm (x^{x-1}(1 - x)^x)^{1/(1-2x)}$$

we get for (½, 0) as origin,

$$y_{-x} = \pm y_x. \tag{29}$$

Hence the discrete real points lie on curves which are symmetrical with respect to both axes and the origin (taken at (½, 0)), while the points themselves and all points of the whole representation are symmetrical with respect to (½, 0), which is therefore a center of symmetry of the configuration.

2. *The Variation of* $|y|$.—Since $y = |(x^{x-1}(x-1)^x)^{1/1-2x}|$ we may differentiate $y = (x^{x-1}(1-x)^x)^{1/1-2x}$ to study the change in the distance from the axis of x of the distribution of points between 0 and 1. For (½, 0) as origin, we find

$$\frac{dy}{dx} = \frac{y}{4x^2}\left(\log\left(\frac{1 - 2x}{1 + 2x}\right) - \frac{4x(4x^2 + 1)}{4x^2 - 1}\right).$$

Developing,

$$\frac{dy}{dx} = \frac{y}{2x^2}\left(\frac{5}{3}(2x)^3 + \frac{9}{5}(2x)^5 + \cdots + \frac{4n+1}{2n+1}(2x)^{2n+1} + \cdots\right), \tag{30}$$

from which it appears that between the limits $-½$ and ½ (0 and 1)

$$\frac{dy}{dx} \gtreqless 0 \quad \text{according as} \quad \begin{matrix} 0 < x < \frac{1}{2} \\ x=0 \\ -\frac{1}{2} < x < 0 \end{matrix}. \tag{31}$$

Hence $|y|$ is decreasing from $x=0$ to $x=1/2$ (original origin) and increasing from $x=1/2$ to $x=1$. At $x=1/2, |y|=2e=5.4$ approximately (see 3) and this value is a minimum. Similarly it can be shown that $|y|$ increases from $-\infty$ to 0, and decreases from 1 to ∞.

3. *The Values of* $|y|$ *when* $x=\pm\infty$, 0, $1/2$, 1.—When $x=\pm\infty$, the value of $|y|$ is easily found to be 0; when $\begin{matrix} x=0 \\ x=1, \end{matrix}$ $|y|=\infty$. These values show that $y=0$, $x=0$ and $x=1$ are asymptotes. When $x=1/2$ put $x=1/2-\alpha$ (α positive or negative). Then

$$y = \left(\frac{2}{1-2\alpha}\right)\left(1+\frac{4\alpha}{1-2\alpha}\right)^{(1-2\alpha)/4\alpha}(-1)^{(1-2\alpha)/4\alpha} \qquad (32)$$

and as $\alpha \doteq 0$, $x \doteq 1/2$,

$$\lim_{x \doteq \frac{1}{2}} |y| = 2e. \qquad (33)$$

4. *The Value Systems Between 0 and 1.**—The value systems between 0 and 1 are of especial interest as here y may have both positive and negative discrete values, but not for the same value of x. Elsewhere this can frequently occur. In fact as necessary for both positive and negative discrete real values, either

$$\frac{x-1}{1-2x} = \pm\frac{1}{2n},$$

whence

$$x = \frac{\pm 2n+1}{\pm 2n+2} \quad \text{and} \quad \frac{x}{1-2x} = \frac{\pm 2n+1}{\mp 2n},$$

or

$$\frac{x}{1-2x} = \pm\frac{1}{2n},$$

whence

$$x = \frac{1}{\pm 2n+1}.$$

These values of x for the upper sign lie between 0 and 1, but give complex values for y (when $n=1$ pure imaginary values). Outside of this range positive and negative real values can occur

* On account of symmetry we need to consider only values between 0 and $1/2$.

together. For single real values between 0 and 1 it is necessary either that

$$\frac{x}{1-2x} = \pm 2n,$$

or that

$$\frac{x}{1-2x} = \pm 2n + 1,$$

whence

$$x = \frac{\pm 2n}{\pm 2n + 1}, \quad \text{or} \quad x = \frac{\pm 2n + 1}{\pm 4n + 3},$$

which values of x for the upper sign lie between 0 and 1, while the other values lie outside of this range. Since

$$y = x^{(x-1)/(1-2x)}(x-1)^{x/(1-2x)},$$

it will contain a factor between 0 and 1.

$$\iota = (-1)^{x/(1-2x)} = \cos\left\{(2n+1)\frac{x}{1-2x}\pi\right\} + i\sin\left\{(2n+1)\frac{x}{1-2x}\pi\right\} = e^{(2n+1)\frac{x}{1-2x}\pi i} \tag{34}$$

(n is any positive or negative integer or zero).

$$\text{Put } (2n+1)\frac{x}{1-2x} = u^* \tag{35}$$

(where if $x < \frac{1}{2}$, $u > 0$; if $x > \frac{1}{2}$, $u < 0$) then

$$x = \frac{u}{2n + 2u + 1}. \tag{36}$$

ι will be equal to $+1$, i, -1, $-i$ according as

$$u = \frac{4p}{2}, \quad \frac{4p+1}{2}, \quad \frac{4p+2}{2}, \quad \frac{4p+3}{2} \tag{37}$$

(p a positive or negative integer or zero) and conversely, and for all other values of u, $\iota = \cos u\pi + i \sin u\pi = e^{u\pi i}$ will be a complex number of modulus 1 containing both the real and imaginary parts, and conversely, while simultaneously y will have

* $1 + (2n+1)\frac{1-2x}{x-1} = u'$ gives the conjugate point (see § 6, 5) and need not therefore be considered in detail.

a value

$$y = +|y|,\ +|y|i,\ -|y|,\ -|y|i,\ |y|e^{u\pi i} \qquad (38)$$

and conversely. From (36) doubly infinitely many values of x can be found giving among them all the different forms for y. Conversely given any value of x between 0 and 1, the character of y can be stated by reference to (36) and (35). Thus from (36) according as u is of the form (37)

$$\begin{aligned}
&(1)\ x = \frac{4p}{2(2n+4p+1)} = \frac{4p}{4r+2} = \frac{2p}{2r+1}, \text{ and gives a value of } y = +|y|,\\
&(2)\ x = \frac{4p+1}{4r} \text{ and gives a value of } y = +|y|i,\\
&(3)\ x = \frac{4p+2}{4r+2} = \frac{2p+1}{2r+1}, \text{ and gives a value of } y = -|y|,\\
&(4)\ x = \frac{4p+3}{4r} \text{ and gives a value of } y = -|y|i.
\end{aligned} \qquad (39)$$

Any other forms for x give complex values only $y = |y|e^{u\pi i}$.

Examples.

1. For $x = \frac{2}{7}$, y is three valued with one real positive value.
2. $x = \frac{4}{7}$ gives $y = 4^3 3^4/7^7$. It has no other values.
3. The formula (36) yields series of given types. Thus for

$$\begin{aligned}
u &= 2,\ n = 0, 1, 2, \cdots,\ x = \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \cdots, \frac{2}{2n+5},\\
u &= 4,\ n = 0, 1, 2, \cdots,\ x = \frac{4}{9}, \frac{4}{11}, \frac{4}{13}, \cdots, \frac{4}{2n+9}.
\end{aligned} \qquad (40)$$

For these values of x we get at least one real positive value each for y and when there are other values they are not real.

$$\begin{aligned}
u &= 1,\ n = 0, 1, 2, \cdots,\ x = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots, \frac{1}{2n+3},\\
u &= 3,\ n = 0, 1, 2, \cdots,\ x = \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \cdots, \frac{1}{2n+7},
\end{aligned} \qquad (41)$$

These values of x give at least a value of y real and negative.

4. If $x = 1/\sqrt{5}$, $u = (2n+1)(\sqrt{5}+2)$ is incommensurable, y is complex always and has an infinite number of values.

5. If $x = \frac{1295}{32657}$, $u = (2n+1)\frac{1295}{30067}$, y has 30067 values, of which one for $n = \frac{30067-1}{2}$ is real and negative ((3) of (39)).

6. For $x = \frac{11}{36}$, we see from ((39)(4)) that y has a value $y = -|y|i$ and there are no real values among the 14 values of y. In this case $u = (2n+1)\frac{11}{14}$; for $n = 3$, $u = \frac{11}{2} = 4 + {}^3$.

In general y is multiple-valued, having from one to an infinite number of values. This may be exhibited as follows: When $x = l/m$, and $m - 2l = \pm 1$ (l and m positive integers),

$$y = \left(\frac{l}{2l \pm 1}\right)^{-1}(-l \mp 1)^{\pm l} = \left(\frac{2l \pm 1}{l}\right)(-l \mp 1)^{\pm l}. \quad (42)$$

Here $x = l/(2l \pm 1)$; there is only one value of y which will be real and positive or negative according as l is even or odd. Thus there will be infinitely many single real positive and infinitely many single real negative values of y. y can not be single-valued when x is negative, nor therefore when x is greater than 1. In general when $x = l/(2l+p)$,

$$y = (-1)^{l/p}\left(\frac{2l+p}{l}\right)\left(1 + \frac{p}{l}\right)^{l/p}, \quad (43)$$

from which it is seen that for $x = l/(2l+p)$ and its symmetrical $x = (l+p)/(2l+p)$ when l and p are relatively prime an infinite number of values of y have p values for $p = 1$, 2, 3, $\cdots$ and that when p is even none of these values are real, and when it is odd one value is real and positive, or negative as l is even or odd. (Outside of 0 and 1 it is possible to have two real values for y equal and opposite in sign.) If $l = 2m$, $p = \infty$, $y = \infty$.

$$\operatorname*{Lim}_{l \doteq \infty} \frac{l+p}{2l+p} = \tfrac{1}{2}.$$

When x is incommensurable y has an infinite number of values all complex unless x lies outside of 0 and 1 when there is always one real value. All the values real and complex lie on a circle of radius $|y|$ for a value of x. All the circles lie on the surface $y = |y|e^{\phi i}$ otherwise expressed in (25) and (26).

5. *The Value Systems Between* $-\infty$ *and 0, and 1 and* ∞.—

For a negative x, $x = -\alpha$,

$$y = -\left(\frac{1}{\alpha}\right)^{\frac{\alpha+1}{2\alpha+1}}\left(\frac{1}{\alpha+1}\right)^{\frac{\alpha}{2\alpha+1}}, \tag{44}$$

hence for every negative value of x there is always a real negative value of y. Also similarly to (35)

$$1 + 2m\frac{x}{1-2x} = -u, \quad \text{or} \quad 1 + 2m\frac{(x-1)}{1-2x} = -u.\text{*} \tag{45}$$

$$x = \frac{-(u+1)}{2(m-u-1)}, \quad \text{or} \quad x = -\frac{(u+1-2m)}{2(m-u-1)} \tag{46}$$

(m an integer, $m - u - 1 > 0$) give formulas from which the value systems can be studied. If $-u = -2p$ (p an integer > 0),

$$x = -\frac{(2p+1)}{2(m-2p-1)}, \quad \text{or} \quad x = -\frac{(2p+1-2m)}{2(m-2p-1)}, \tag{47}$$

If in (46) $m = 0$, $u = -1$, x becomes indeterminate. If $p = 0$,

$$x = -\frac{1}{2(m-1)}, \quad \text{or} \quad x = +\frac{(2m-1)}{2(m-1)},$$

If $p > 0$, $m = 1$,

$$x = \frac{+(2p+1)}{4p}, \quad \text{or} \quad x = \frac{2p-1}{4p},$$

If $m = 2$,

$$x = -\frac{(2p+1)}{2(1-p)}, \quad \text{or} \quad x = \tfrac{1}{2}.$$

The last five values of x must be excluded since they do not satisfy $m - 2p - 1 > 0$, and give positive values for x. For

$$x = \frac{-(2p+1)}{2(m-2p-1)},$$

$$y = \left(\frac{2(m-2p-1)}{-2p-1+2m}\right)^{\frac{2p+1}{2(m+p)}}\left(\frac{2(m-2p-1)}{2p+1}\right)^{\frac{2p+1-2m}{2(m+p)}}(-1)^{\frac{m}{m+p}}. \tag{48}$$

* $1 + 2m\,\frac{(x-1)}{1-2x} = u'$ gives the conjugate point (see (62)).

If $m+p=2r+1$, y has both a positive and negative real value at least. If $p=2q$, $m=2s+1$,

$$x=-\frac{(2p+1)}{2(m-2p-1)}=\frac{-(4q+3)}{4(s-2q)}, \tag{49}$$

and if $p=2q+1$, $m=2s$,

$$x=-\frac{(4q+3)}{2(2s-2q-3)}. \tag{50}$$

For $x>1$, symmetry requires that there shall be in addition to the continuous positive real value of y also negative values for special values of x, as for

$$x=1+\frac{2p+1}{2(m-2p-1)}=\frac{2m-2p-1}{2(m-2p-1)},$$

III. *The Sequence and Contiguity of the Real Discrete Points.*

Some examples of the method of proving that between two discrete real positive or discrete real negative points, however close, an infinity of real and complex points still lies, will be given. It is only necessary to prove this of the positive discrete values between $-\infty$ and 0, and between 0 and 1. Symmetry will show that it is true of the negative discrete values between 1 and ∞ and between 0 and 1. Similar relations of contiguity exist for the complex points. From $-\infty$ to 0, positive discrete values exist for

$$x=-\frac{(2p+1)}{2(m-2p-1)}$$

when $m+p$ is odd by (48). Put $q=2p+1$, $t=m-2p-1$. Then $x=-(q/2t)$ gives a positive value for y. Then

$$x=-\frac{(lq\pm 2r)}{2lt},$$

$l=2s+1$, r an integer, gives a positive real point for y since

$$\begin{aligned} x &= -\frac{lq\pm 2r}{2lt}=-\frac{(2pl+l\pm 2r)}{2(lm-2lp-l)} \\ &= -\frac{(2(pl\pm r+s)+1)}{2((lm\pm 2r)-2(lp\pm r+s)-1)}=-\frac{(2(p'+1))}{2(m'-2p'-1)}, \end{aligned} \tag{51}$$

and $m' + p'$ is odd, if r and s are odd, or both even. Also

$$-\frac{lq+2r}{2lt} < -\frac{q}{2t} < -\frac{lq-2r}{2lt} \tag{52}$$

As $l \doteq \infty$

$$-\frac{(lq \pm 2r)}{2lt} \doteq -\frac{q}{2t}.$$

Moreover we may take $r = 1, 2, 3 \cdots$. (For every $x < 0$, there is always one real negative point and generally complex points. Incommensurable values of $x < 0$ and $x > 1$ always give one real point for y. Between 0 and 1 they give no real points.) Hence the statement for points between $-\infty$ and 0. Symmetry shows the same relation for the negative discrete points for $x > 1$.

From 0 to 1 by (36), if $u = 2p$, $q = 2n + 2u + 1$, $x = 2p/q$ and this by (1) of (39) gives a positive y. Similarly

$$x' = \frac{2lq}{lq \pm (2r+1)}, \; l = 2m, \tag{53}$$

also gives a positive y, and

$$\frac{2lp}{lq-(2r+1)} > \frac{2p}{q} > \frac{2lp}{lq+(2r+1)}. \tag{54}$$

As $l \doteq \infty$,

$$\frac{2lp}{lq \pm (2r+1)} \doteq \frac{2p}{q}, \quad \text{or} \quad x' \doteq x.$$

We might have taken

$$x' = \frac{2lp}{lq \pm 2r}$$

with l odd. Since r may be taken as $0, 1, 2, \cdots$, it follows that an infinite number of real positive points lie infinitely near to $x = 2p/q$.

In like manner

$$x = \frac{\frac{4p+1}{2}}{2n+4p+2}$$

gives a value of $y = +|y|i$.

$$x' = \frac{\frac{4lp \pm 1}{2}}{lq \pm (2r+1)}$$

if l is odd gives a value of $y = \pm|y|i$, where

$$\lim_{l \doteq \infty} x' = \frac{2p}{q}$$

(which gives a positive value of y) and where

$$\frac{\frac{4lp-1}{2}}{lq-m} < \frac{2p}{q} < \frac{\frac{4lp+1}{2}}{lq+m} \quad m = 2r+1 \text{ and } q + 4pm > 0. \quad (55)$$

Also

$$x = \frac{\frac{4lp-s-l}{2}}{lq-l\pm s}, \quad l = 4r-1,\ s = 4t+1$$

gives a positive y and its limit as $l \doteq \infty$ is

$$\frac{\frac{4p-1}{2}}{q-1},$$

for which y has a value $y = -|y|i$.

Similarly

$$x = \frac{4lp-s-l}{lq-l}, \quad l = 4r, \quad s = 4t+1,$$

gives a value of $y = -|y|i$, and its limit as $l \doteq \infty$ is

$$\frac{\frac{4p-1}{2}}{q-1}.$$

Now,

$$\underset{(y=-|y|i)}{\frac{\frac{4p-1}{2}}{q-1}} > \underset{(y'=+|y'|)}{\frac{\frac{4lp-s-l}{2}}{lq-l-s}} > \underset{(y''=+|y''|)}{\frac{\frac{4lp-s-l}{2}}{lq-l+s}} \quad \text{where} \quad \begin{array}{l} l = 4r-1, \\ s = 4t+1, \\ q > 4p, \end{array} \quad (56)$$

$$\frac{\frac{4l'p-s-l'}{2}}{l'q-l'} < \frac{\frac{4lp-s-l}{2}}{lq-l-s}, \quad \underset{(l'=4k)}{\frac{l'(q-4p)+s}{q-1}} < l,$$

hence

$$\begin{array}{ccccc} x & & x' & & x'' \\ (y=-|y|i) & & (y'=+|y'|) & & (y''=-|y''|i) \\ \dfrac{\dfrac{4p-1}{2}}{q-1} & > & \dfrac{\dfrac{4lp-s-l}{2}}{lq-l-s} & > & \dfrac{\dfrac{4l'p-s-l'}{2}}{l'q-l'}. \end{array} \tag{57}$$

As $l=\infty$, Lim $x''=x$ and however near x'' is to x, both giving values of y ($y=-|y|i$, $y''=-|y''|i$) purely negative imaginary, there is a value x', for which $y'=+|y'|$, and whose limit is also x, and $s=1, 5, 9, \cdots$. In like manner

$$\begin{array}{ccccc} x & & x' & & x'' \\ (y=+|y|) & & (y'=|y'|i) & & (y''=+|y''|) \\ \dfrac{2p}{q} & < & \dfrac{\dfrac{4lp+s}{2}}{lq+s} & < & \dfrac{2l'p}{l'q-s} \end{array} \quad \text{where} \quad \begin{array}{l} l=2r+1,\ l'=2k \\ s=4t+1 \end{array} \tag{58}$$

and

$$\frac{1}{4p}(l'(q-4p)-s)<l.$$

Hence however near the real positive point y'' is to the real positive point y by taking l' sufficiently large, an infinite number of purely imaginary points y' lie between by taking

$$l>\tfrac{1}{4}(l'(q-4p)-s).$$

Not only may l, l' be varied but $s=1, 5, 9, \cdots$. If x' is incommensurable there are infinitely many such points between x and x'' whose y's, y and y'', have positive real points. For

$$\begin{array}{ccccc} x & & x' & & x'' \\ \dfrac{2p}{q} & < & \dfrac{2p}{q}+\eta & < & \dfrac{2lp}{lq-s} \end{array} \quad \text{where} \quad \begin{array}{l} l=2r,\ s=2t+1 \\ \text{or } l=2r+1,\ s=2t, \end{array} \eta<\frac{2p}{q}\frac{s}{lq-s} \tag{59}$$

and where as η incommensurable $\doteq 0$, $y'=|y'|e^{u\pi i}$, lies between two positive points ($s=1, 3, 5\cdots$ or $2, 4, 6\cdots$) however near, as $l=\infty$, they may be.

IV. *The Real Curves Passing through the Discrete Real Points Between o and 1.*—Since for the real positive points $u=2m$, the real positive discrete points between 0 and 1 satisfy the equation $y=(x^{x-1}(1-x)^{x})^{1/1-2x}$ which has a real curve between 0 and 1, and discrete negative points between the same

limits and discrete positive and negative points elsewhere, and has spirals through its discrete and other real points, all of which spirals and multiple-valued points lie on the surface (25), and whose discrete points lie on the real parts of the curves $y = \pm(x^{x-1}(x-1)^x)^{1/1-2x}$. Similarly the negative discrete points between 0 and 1 of our function lie on the real part of the curve $y = -(x^{x-1}(1-x)^x)^{1/1-2x}$.

V. *The Spiral Systems.*

Since

$$\frac{x-1}{1-2x} + \frac{x}{1-2x} \equiv -1,$$

y between $-\infty$ and 0 can be expressed either as

$$y_1 = \frac{1}{x}\left(\frac{x-1}{x}\right)^{\frac{x}{1-2x}} = \frac{1}{\alpha}\left(\frac{\alpha+1}{\alpha}\right)^{\frac{x}{1-2x}} e^{v\pi i},$$

where

$$v = 1 + 2m\frac{x}{1-2x} \tag{60}$$

$(x = -\alpha)$, or as

$$y_2 = \frac{1}{x-1}\left(\frac{x}{x-1}\right)^{\frac{x-1}{1-2x}} = \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\frac{x-1}{1-2x}} e^{v'\pi i},$$

where

$$v' = 1 + 2m\frac{x-1}{1-2x}. \tag{61}$$

(See also (45).)

Since $v + v' = -2(m-1)$, $v'\pi = -2(m-1)\pi - v\pi$, and

$$y_1 = |y|e^{v\pi i} \text{ and } y_2 = |y|e^{-v\pi i} \text{ are conjugates.} \tag{62}$$

By § 3 y_1 and y_2 are families of right- and left-handed spirals respectively when $m > 0$.* The spirals are conjugate spirals which pierce the real plane and intersect in positive or negative real points. Symmetry shows the same thing between 1 and ∞.

* When $m < 0$ this is reversed. When $m > 0$, v increases with x, since $D_x v = \frac{2m}{(1-2x)^2}$.

Between 0 and 1 y can be expressed either as

$$y_1 = \frac{1}{x}\left(\frac{1-x}{x}\right)^{\frac{x}{1-2x}} e^{u\pi i}, \quad u = (2n+1)\frac{x}{1-2x} \tag{63}$$

or as

$$y_2 = \frac{1}{1-x}\left(\frac{x}{1-x}\right)^{\frac{x-1}{1-2x}} e^{u'\pi i}, \quad u' = 1 + (2n+1)\frac{x-1}{1-2x}. \tag{64}$$

As

$$u + u' = -2n, \quad e^{u'\pi i} = e^{-u\pi i},$$

hence

$$y_1 = |y|e^{u\pi i} \text{ and } y_2 = |y|e^{-u\pi i} \text{ are conjugates.} \tag{65}$$

y_1 and y_2 show y as families of right- and left-handed conjugate spirals, piercing the real plane and intersecting each other in real discrete points. ($n \geqq 0$. If $n < 0$ this is reversed.)

Thus from $-\infty$ to ∞ conjugate right- and left-handed spirals complete the representation and pierce the real plane in the discrete or other real points. The discrete complex points on the circle at x, finite or infinite in number as x is commensurable or incommensurable by (43), all lie in the surface $y = |y|e^{\phi i}$. As x advances we see how it is that the distribution of points on the circles presents an infinitely complex kaleidoscopic appearance and disappearance of points in the surface of revolution, but in such a way that the conjugate spirals are generated throughout. We consider especially the spirals between 0 and 1. The change in x for a change in u is

$$\Delta x = \frac{(1-2x)^2 \Delta u}{2n + 1 + 2(1-2x)\Delta u}, \tag{66}$$

since by (65) it is only necessary to consider one of the conjugate families, and furthermore we need only consider this one between 0 and $\frac{1}{2}$. If $\Delta u = 2$, the moving point will have made a complete turn on its spiral about OX. If $x = \frac{2}{5}$ $(n = 0)$, and if $\Delta u = 2$, the change in x necessary for one complete turn and return of y to a positive real point is $\Delta x = \frac{2}{45}$. The point x arrived at is $x = \frac{2}{5} + \frac{2}{45} = \frac{4}{9}$, which is in fact seen to have a positive value for y.

$$\Delta u = \frac{(2n+1)\Delta x}{(1-2x)(1-2x-2\Delta x)}, \tag{67}$$

Between 0 and 1

$$x = \frac{u}{2n + 2u + 1}$$

and hence

$$x' = x + \Delta x = \frac{u + \Delta u}{2n + 2u + 1 + 2\,\Delta u} \tag{68}$$

gives points on the same spiral for a particular value of the parameter n. Let S_n denote a particular spiral of the family for a particular value of n. Corresponding to n is a point (x, y_n) which lies on S_n. When y is single-valued, (x, y) lies on every S_n or every S_n goes through (x, y) and corresponding to every value of n is (x, y). Thus when $x = 1/3$, y is single-valued and negative real, for

$$u = 2n + 1\,\frac{\frac{1}{3}}{1 - \frac{2}{3}} = 2n + 1,$$

and $e^{(2n+1)\pi i} = -1$, and $y = -6$. $(1/3, -6)$ belongs to every spiral of the family, that is, infinitely many conjugate spirals go through it. For $x = 1/5$, y is 3-valued, Fig. 2.

$$u = (2n + 1)\tfrac{1}{3}, \quad x = \frac{u}{2n + 2n + 1} = \frac{(2n + 1)\frac{1}{3}}{2n + 1 + 2(2n + 1)\frac{1}{3}},$$

whence

$$\left.\begin{aligned} x &= \underset{n=1}{\tfrac{1}{5}} = \underset{n=4}{\tfrac{3}{15}} = \underset{n=7}{\tfrac{5}{25}} = \text{etc.}, \quad u \equiv 1 \pmod 2 \\ &= \underset{n=2}{\frac{\frac{5}{3}}{5+\frac{10}{3}}} = \underset{n=5}{\frac{\frac{11}{3}}{11+\frac{22}{3}}} = \underset{n=8}{\frac{\frac{17}{3}}{17+\frac{34}{3}}} = \text{etc.}, \quad u \equiv -\tfrac{1}{3} \pmod 2 \\ &= \underset{n=3}{\frac{\frac{7}{3}}{7+\frac{14}{3}}} = \underset{n=6}{\frac{\frac{13}{3}}{13+\frac{26}{3}}} = \frac{\frac{19}{3}}{19+\frac{38}{3}} = \text{etc.}, \quad u \equiv \tfrac{1}{3} \pmod 2 \end{aligned}\right\} \tag{69}$$

In this case

$$\left.\begin{aligned} &(\tfrac{1}{5}, -|y|) \text{ belongs to the spirals } S_{3n+1}, \\ &(\tfrac{1}{5}, |y|e^{-(\pi/3)i}) \text{ belongs to the spirals } S_{3n+2}, \\ &(\tfrac{1}{5}, |y|e^{(\pi/3)i}) \text{ belongs to the spirals } S_{3n}. \end{aligned}\right\} \tag{70}$$

But

$$\begin{pmatrix}\text{The groups} \\ S_{3n},\ S_{3n+1},\ S_{3n+2}\end{pmatrix} = \begin{pmatrix}\text{The group} \\ S_n\end{pmatrix}, \quad n = 0, \pm 1, \pm 2, \cdots. \tag{71}$$

Hence at $x = 1/5$ *all the spirals* pass through the circle at $x = 1/5$ in three groups through 3 points on the circle.

For $x = \frac{1}{7}$, y is 5-valued.

$$\left.\begin{array}{ll}
\left(\tfrac{1}{7}, -|y|\right)\begin{array}{l} n = 2, 7, 12, \cdots \\ u = 1, 3, 5, \cdots \end{array} & \text{belongs to } S_{5n+2} \\[1ex]
\left(\tfrac{1}{7}, |y|e^{3\pi i/5}\right)\begin{array}{l} n = 1, 6, 11, \cdots \\ u = \tfrac{3}{5}, \tfrac{13}{5}, \tfrac{23}{5}, \cdots \end{array} & \text{belongs to } S_{5n+1} \\[1ex]
\left(\tfrac{1}{7}, |y|e^{\frac{7}{5}\pi i}\right)\begin{array}{l} n = 3, 8, \cdots \\ u = \tfrac{7}{5}, \tfrac{17}{5}, \cdots \end{array} & \text{belongs to } S_{5n+3} \\[1ex]
\left(\tfrac{1}{7}, |y|e^{\frac{9}{5}\pi i}\right)\begin{array}{l} n = 4, 9, \cdots \\ u = \tfrac{9}{5}, \tfrac{19}{5}, \cdots \end{array} & \text{belongs to } S_{5n+4} \\[1ex]
\left(\tfrac{1}{7}, |y|e^{\frac{1}{5}\pi i}\right)\begin{array}{l} n = 0, 5, \cdots \\ u = \tfrac{1}{5}, \tfrac{11}{5}, \cdots \end{array} & \text{belongs to } S_{5n}
\end{array}\right\} \tag{72}$$

But $(S_{5n}, S_{5n+1}, S_{5n+2}, S_{5n+3}, S_{5n+4})$

$$= (S_n), \quad n = 0, \pm 1, \pm 2, \cdots. \tag{73}$$

Hence at $x = \frac{1}{7}$ *all the infinitely many spirals* pass in 5 groups through 5 points on the circle at $x = \frac{1}{7}$, whose radius is $|y|$. When x is incommensurable y has a different value for each value of n; thus y has an infinite number of values, and to each point (x, y_n) belongs only one spiral S_n. That is

$$\left.\begin{array}{l}
(x, y_0) \text{ belongs to } S_0, \\
(x, y_1) \text{ belongs to } S_1, \\
(x, y_2) \text{ belongs to } S_2, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
(x, y_{-1}) \text{ belongs to } S_{-1}, \\
(x, y_{-2}) \text{ belongs to } S_{-2}. \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{array}\right\} \tag{74}$$

Here there is a one-to-one correspondence between the spiral and the point through which it passes. Through each point on the circle representing a value of y passes one and only one spiral. In the neighborhood of $x = \frac{1}{2}$, the spirals become infinitely compressed, since the number of turns becomes infinitely great as $x \doteq \frac{1}{2}$.

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